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Graphs with characteristic-dependent well-covered dimension

Joseph Burdick

Abstract. The dimension of the well-covered space of certain graphs depends upon characteristic of the field of scalars of the vector space. We investigate graphs that have this characteristic-dependent well-covered dimension and show how more of these graphs can be constructed.

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1 Introduction

The well-covered dimension is a parameter of a graph that can sometimes depend upon the characteristic of the field considered when determining it. Our objective is to find the smallest graphs for which the well-covered dimension depends upon the characteristic of the field used to compute this parameter. In order to define well-covered dimension, we must first define the ideas of maximal independent sets and well-covered weightings. In this paper, all graphs will be finite, simple and undirected. For a graph G, let $V(G)$ denote the set of G's vertices.

Definition 1. Let G be a graph. A subset M of $V(G)$ is an **independent set** of G if no pair of vertices in M are adjacent (there is no edge connecting them). An independent set M of G is said to be **maximal** if it is not properly contained in any other independent set of G.

Definition 2. Let $\mathbb{F}$ be a field and G a graph. A **well-covered weighting** of G is a function $w : V(G) \to \mathbb{F}$ such that $\sum_{v \in M} w(v)$ is constant for all maximal independent sets M of G.

There is an abundant amount of research about graphs in which every maximal independent set has the same cardinality; these graphs are said to be well-covered (see Plummer [8, 9]). Note that a well-covered graph may be also defined as a graph that admits all constant functions as well-covered weightings. Hence, the study of well-covered weightings is an attempt to generalize the study of well-covered graphs.

Definition 3. Let G be a graph and $\mathbb{F}$ a field. The $\mathbb{F}$-vector space of all well-covered weightings of G is called the **well-covered space** of G (relative to $\mathbb{F}$). The **well-covered dimension** of G over $\mathbb{F}$ is the dimension of the well-covered vector space of G. We will denote this number by $wcdim(G, \mathbb{F})$.

The concept of the well-covered dimension of a graph was first studied by Caro, Ellingham, Ramey, and Yuster [4, 5], and was later developed further by Brown and Nowakowski [3] in 2005. Some more work has been done on this topic recently; see Birnbaum et al. [1], and Clemente [6] for new results only involving graph theory, and see Hauschild, Ortiz and Vega [7] for an attempt to apply the well-covered dimension of a graph to the study of point-line configurations. We refer the reader to Bondy and Murty's book [2] for a more complete introduction to concepts in graph theory.

The technique we will use to compute the well-covered dimension of a given graph is fairly natural. Assuming that we know that $M_1, M_2, ..., M_k$ are all the maximal independent sets of G, then the well-covered weightings of G are given by the solutions to the linear system of equations:

$$\begin{gathered}\sum_{v\in M_1} x_v = \sum_{v\in M_2} x_v \\ \sum_{v\in M_1} x_v = \sum_{v\in M_3} x_v \\ \vdots \\ \sum_{v\in M_1} x_v = \sum_{v\in M_k} x_v\end{gathered}$$

where x_v denotes the weight of $v \in V(G)$ under a weighting. This system may be represented by a matrix, which we call the **associated matrix** of G and denote by A_G. Because there are $k-1$ equations in this system, and because every one of G's vertices is contained in at least one maximal independent set, we can see that A_G is a $(k-1)\times n$ matrix where n is the number of G's vertices. The well-covered weightings of G form the null space of A_G, so the well-covered dimension of G can be expressed as

$$wcdim(G,\mathbb{F}) = |V(G)| - rank(A_G).$$

Note that depending on the characteristic of the field $\mathbb{F}$, $rank(A_G)$ could vary. So, for some graphs the well-covered dimension is characteristic-dependent. This is the property we intend to investigate in this article.

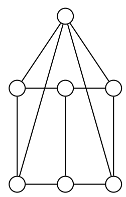
Not much is known about why some graphs have this property. What it is known is that for any given prime number p there are graphs G such that $wcdim(G,\mathbb{F}) \neq wcdim(G,\mathbb{Q})$, where $char(\mathbb{F}) = p$ and $\mathbb{Q}$ is the field of rationals, with characteristic 0. This was first shown by Brown and Nowakowski [3]. We say that these graphs have characteristic-dependent well-covered dimension. Birnbaum et al. [1] show that crown graphs may be used to construct such graphs, which are smaller in order (number of vertices) than those given in Brown and Nowakowski [3], but are still not the smallest possible graphs (by order) with this property.

Our focus is, given a prime number p, to find the smallest (by order) graph G such that $wcdim(G,\mathbb{F}) \neq wcdim(G,\mathbb{Q})$ for fields with $char(\mathbb{F}) = p$. In Section 2, we consider several examples of graphs with characteristic-dependent well-covered dimension and demonstrate how the well-covered dimension of a graph can be computed. In Section 3, we show how graphs with characteristic-dependent well-covered dimension can be constructed for any given characteristic.

2 Smallest graphs with characteristic-dependent well-covered dimension

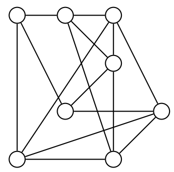
In order to find the smallest possible graphs with characteristic-dependent well-covered dimension, we wrote a program in Sage [10] to calculate the well-covered dimension of all graphs of order less than or equal to 10. In this way we found the smallest graphs with characteristic-dependent well-covered dimension so that $wcdim(G, \mathbb{F}) \neq wcdim(G, \mathbb{Q})$ for fields with characteristics 2, 3, and 5, respectively. They are:

G_7:



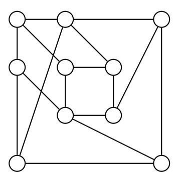
$$wcdim(G_7, \mathbb{F}) = \begin{cases} 2 & \text{if } char(\mathbb{F}) \neq 2 \\ 3 & \text{if } char(\mathbb{F}) = 2 \end{cases}$$

G_8:



$$wcdim(G_8, \mathbb{F}) = \begin{cases} 1 & \text{if } char(\mathbb{F}) \neq 3 \\ 2 & \text{if } char(\mathbb{F}) = 3 \end{cases}$$

G_{10}:



$$wcdim(G_{10}, \mathbb{F}) = \begin{cases} 0 & \text{if } char(\mathbb{F}) \neq 5 \\ 1 & \text{if } char(\mathbb{F}) = 5 \end{cases}$$

We also searched for graphs with order 11 that had characteristic-dependent well-covered dimension. We estimated that this search would take about two weeks on the machine we were running on, and so we only obtained partial results for graphs on eleven vertices. We would like to mention that the graphs listed above are not the only ones having characteristic-dependent well-covered dimension. In fact, the analysis of all those graphs helped us to conjecture many of the results in the following section.

3 Construction of graphs with characteristic-dependent well-covered dimension

Let G be a graph, and let $E(G)$ denote the set of edges in G. For $v \in V(G)$ we define:

$$N(v) = \{w \in V(G);\ vw \in E(G)\}$$

.

If $W \subseteq V(G)$, we denote by $G[W]$ the induced subgraph of G, formed by all of the vertices in W and all edges in $E(G)$ connecting pairs of vertices in W. We define K_n to be the complete graph, the graph with n vertices and each vertex adjacent to every other vertex. A clique C is a subset of $V(G)$ such that $G[C]$ is a complete graph. We will say that a vertex is incident with an edge if that edge connects to that vertex. We will also say that we contract an edge of a graph if we remove the edge from the graph and merge the two vertices that are incident with the edge.

Lemma 1. *Let G be a graph.*

1. *Assume that G contains a clique C such that $N(v)\cup\{v\} = N(w)\cup\{w\}$, for all $v, w \in C$. Then, $wcdim(G) = wcdim(G')$, where G' is the graph obtained after contracting all the edges in $G[C]$.*

2. *For any given $v \in G$ and $n \in \mathbb{N}$ define $G_{v,n}$ as the graph with*

   $$V(G_{v,n}) = (V(G) \setminus \{v\}) \cup V(K_n)$$
   $$E(G_{v,n}) = (E(G) \setminus \{e \in E(G);\ v \text{ is incident with } e\}) \cup \{ab;\ a \in N(v),\ b \in K_n\}.$$

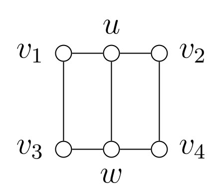
   Then, $wcdim(G) = wcdim(G_{v,n})$.

Proof. (1) If M is a maximal independent set of G containing $v \in C$, then we can create a new maximal independent set of G by replacing v by any other element $w \in C$ and leaving all other elements in M intact. Hence, when all the vertices in $G[C]$ are contracted the well-covered dimension does not change.
(2) This follows directly from (1), as $C = V(K_n)$ is a clique of $G_{v,n}$ satisfying the hypothesis of part (1), and G is the graph obtained from $G_{v,n}$ after contracting K_n to a vertex. $\square$

Remark 1. We can now take any of the graphs in Section 2 and use Lemma 1 to construct graphs with characteristic-dependent well-covered dimension of any order we may want. For example, using G_7 we can prove that for every $n \geq 7$ there is a graph G of order n for which $wcdim(G, \mathbb{F})$ takes different values depending on whether $char(\mathbb{F})$ is 2 or not.

We improve part of the previous remark in the following theorem.

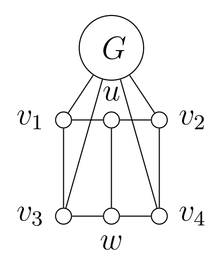
Theorem 1. *Let G_{K_2} be the graph below.*



Let G be a non-empty graph, and H_G be the disjoint union of G_{K_2} and G with added edges $v_i x$, for all $x \in V(G)$ and $i = 1, 2, 3, 4$. Then,

$$wcdim(H_G, \mathbb{F}) = \begin{cases} wcdim(G, \mathbb{F}) + 1 & \text{if } char(\mathbb{F}) \neq 2 \\ wcdim(G, \mathbb{F}) + 2 & \text{if } char(\mathbb{F}) = 2 \end{cases}$$

Proof. Let G be a non-empty graph of order m where $V(G) = \{x_1, \ldots, x_m\}$. We represent H_G as in the figure below,



We notice that the maximal independent sets of H_G have the form

$$\{v_1, v_4\}, \{v_2, v_3\}, \{v_1, v_2, w\}, \{v_3, v_4, u\}, \{u\} \cup M_i, \{w\} \cup M_i$$

where M_i is a maximal independent set of G.

We let n be the number of maximal independent sets of G, and for any maximal independent set, M_i, of G we define R_{M_i} to be the $m \times 1$ matrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_m \end{pmatrix}$$

where $a_j = -1$ if $x_j \in M_i$ and $a_j = 0$ otherwise. Using $\mathbf{0}_m$ for the $m \times 1$ matrix with all 0 entries, we get that the associated matrix of H_G is:

$$\begin{pmatrix}
1 & -1 & -1 & 1 & 0 & 0 & \mathbf{0}_m \\
0 & -1 & 0 & 1 & 0 & -1 & \mathbf{0}_m \\
1 & 0 & -1 & 0 & -1 & 0 & \mathbf{0}_m \\
1 & 0 & 0 & 1 & -1 & 0 & R_{M_1} \\
1 & 0 & 0 & 1 & 0 & -1 & R_{M_1} \\
1 & 0 & 0 & 1 & -1 & 0 & R_{M_2} \\
1 & 0 & 0 & 1 & 0 & -1 & R_{M_2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 1 & -1 & 0 & R_{M_n} \\
1 & 0 & 0 & 1 & 0 & -1 & R_{M_n}
\end{pmatrix}$$

By performing row-reduction, we obtain the Hermite Normal form of A_{H_G}:

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & R_{M_1} \\
0 & 1 & 0 & -1 & 0 & 1 & \mathbf{0}_m \\
0 & 0 & 1 & 1 & 0 & 0 & R_{M_1} \\
0 & 0 & 0 & 0 & 1 & 1 & \mathbf{0}_m \\
0 & 0 & 0 & 0 & 0 & 2 & \mathbf{0}_m \\
0 & 0 & 0 & 0 & 0 & 0 & M_n - M_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & M_n - M_{n-1}
\end{pmatrix}$$

We can see that the fifth row of this matrix will have only 0 entries over a field with characteristic 2, and that the rows below are equivalent to the associated matrix A_G. Hence, the claimed formula for $wcdim(H_G, \mathbb{F})$ is proven. $\square$

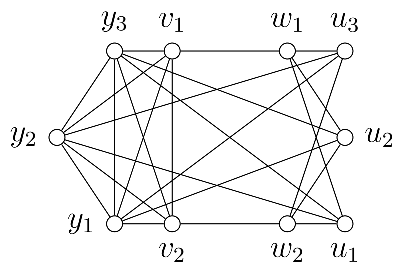
We now take a look at graphs that will have characteristic-dependent well-covered dimension for odd primes.

Let $K_{n,m}$ denote a complete bipartite graph, where the vertices of $K_{n,m}$ can be partitioned into two partite subsets, V with cardinality n and W with cardinality m, such that no two vertices in V are adjacent, no two vertices in W are adjacent, and every $v \in V$ is adjacent to every $w \in W$.

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 2$. There exists a graph G on $2n+4$ vertices such that*

$$wcdim(G,\mathbb{F}) = \begin{cases} 1 & \text{if } char(\mathbb{F}) \nmid (2n-1) \\ 2 & \text{if } char(\mathbb{F}) \mid (2n-1) \end{cases}$$

Proof. Let G_n be the graph constructed from the disjoint union of K_{n+2} and $K_{2,n}$, where $V(K_{n+2}) = \{y_1, y_2, \ldots, y_n, v_1, v_2\}$, $V(K_{2,n}) = \{w_1, w_2, u_1, u_2, \ldots, u_n\}$, and $\{w_1, w_2\}$ is the 2-vertex partite set of $K_{2,n}$. To the edges already present in K_{n+2} and $K_{n,2}$ we add the edges v_1w_1, v_2w_2, and y_iu_j, for all $i \neq j \in \{1, \ldots, n-2\}$.
The following figure shows what G_3 looks like.



It is easy to see that G_n has exactly $2n+4$ maximal independent sets. They are:

(i) $M_i = \{y_i, w_1, w_2\}$, for all $i \in \{1, \ldots, n\}$,

(ii) $N_i = \{y_i, u_i\}$, for all $i \in \{1, \ldots, n\}$,

(iii) $L_1 = \{v_1, w_2\}$,

(iv) $L_2 = \{v_2, w_1\}$,

(v) $K_1 = \{v_1, u_1, \ldots, u_n\}$, and

(vi) $K_2 = \{v_2, u_1, \ldots, u_n\}$.

These maximal independent sets yield the following $2n+3$ linear equations:

$$\sum\nolimits_{v \in M_i} x_v = \sum\nolimits_{v \in M_{i+1}} x_v \qquad\qquad \sum\nolimits_{v \in N_i} x_v = \sum\nolimits_{v \in N_{i+1}} x_v$$

for $i = 1, \ldots, n-1$, and

$$\begin{aligned} \sum\nolimits_{v \in M_n} x_v &= \sum\nolimits_{v \in N_1} x_v \\ \sum\nolimits_{v \in N_n} x_v &= \sum\nolimits_{v \in L_1} x_v \\ \sum\nolimits_{v \in L_1} x_v &= \sum\nolimits_{v \in L_2} x_v \\ \sum\nolimits_{v \in L_2} x_v &= \sum\nolimits_{v \in K_1} x_v \\ \sum\nolimits_{v \in K_1} x_v &= \sum\nolimits_{v \in K_2} x_v \end{aligned}$$

where x_v denotes the weight of $v \in V(G)$ under a weighting.

In order to describe A_{G_n} we need some notation. We will use $M_1 - M_2$ to denote the row given by the equation $\sum_{v \in M_1} x_v - \sum_{v \in M_2} x_v = 0$. We will follow a similar notation for all other rows. Also, we let M be the following $(n-1) \times n$ matrix

$$
M = \begin{pmatrix}
1 & -1 & 0 & \dots & 0 & 0 \\
0 & 1 & -1 & \dots & 0 & 0 \\
\vdots & & \ddots & \ddots & & \vdots \\
0 & 0 & 0 & \dots & -1 & 0 \\
0 & 0 & 0 & \dots & 1 & -1
\end{pmatrix}
$$

Finally, we would like to clarify to the reader that we are placing labels around the matrix. They are there to show where each entry in the matrix comes from. We get:

$$
\begin{array}{rc}
 & \begin{array}{ccc|ccc|cccc} y_1 & \dots & y_n & u_1 & \dots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r} M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ \hline N_n - L_1 \\ \hline M_n - N_1 \\ \hline N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2 \end{array} &
\left(\begin{array}{ccc|ccc|cccc}
 & & & & & & & & & \\
 & M & & & 0 & & & & 0 & \\
 & & & & & & & & & \\ \hline
0 & \mathbf{0}_{n-2} & 1 & 0 & \mathbf{0}_{n-2} & 1 & -1 & 0 & 0 & -1 \\ \hline
-1 & \mathbf{0}_{n-2} & 1 & -1 & \mathbf{0}_{n-2} & 0 & 0 & 0 & 1 & 1 \\ \hline
 & & & & & & & & & \\
 & M & & & M & & & & 0 & \\
 & & & & & & & & & \\ \hline
 & & & & \mathbf{0}_n & & 1 & -1 & -1 & 1 \\
 & 0 & & & (-1)_n & & -1 & 1 & 1 & 0 \\
 & & & & \mathbf{0}_n & & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}
$$

We now subtract the first row-block from the fourth. We also notice that the sum of the rows in M add up to $(1, \mathbf{0}_{n-2}, -1)$. So, we add all the rows in the first row-block to the row

$M_n - N_1$, and we get

$$
\begin{array}{r@{\;}c}
 & \begin{array}{ccc|ccc|cccc} y_1 & \cdots & y_n & u_1 & \cdots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r} M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ \hline N_n - L_1 \\ \hline M_n - N_1 \\ \hline N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2 \end{array} &
\left(\begin{array}{ccc|ccc|cccc}
 & & & & & & & & & \\
 & M & & & 0 & & \multicolumn{4}{c}{0} \\
 & & & & & & & & & \\ \hline
0 & \mathbf{0}_{n-2} & 1 & 0 & \mathbf{0}_{n-2} & 1 & -1 & 0 & 0 & -1 \\ \hline
 & \mathbf{0}_n & & -1 & \mathbf{0}_{n-2} & 0 & 0 & 0 & 1 & 1 \\ \hline
 & & & & & & & & & \\
 & 0 & & & M & & \multicolumn{4}{c}{0} \\
 & & & & & & & & & \\ \hline
 & & & & \mathbf{0}_n & & 1 & -1 & -1 & 1 \\
 & 0 & & & (-1)_n & & -1 & 1 & 1 & 0 \\
 & & & & \mathbf{0}_n & & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}
$$

Since the sum of the rows in M add up to $(1, \mathbf{0}_{n-2}, -1)$, we add the rows $N_1 - N_2$ through $N_{n-1} - N_n$ and the row $M_n - N_1$ to the row $N_n - L_1$. We get

$$
\begin{array}{r@{\;}c}
 & \begin{array}{ccc|ccc|cccc} y_1 & \cdots & y_n & u_1 & \cdots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r} M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ \hline N_n - L_1 \\ \hline M_n - N_1 \\ \hline N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2 \end{array} &
\left(\begin{array}{ccc|ccc|cccc}
 & & & & & & & & & \\
 & M & & & 0 & & \multicolumn{4}{c}{0} \\
 & & & & & & & & & \\ \hline
0 & \mathbf{0}_{n-2} & 1 & & \mathbf{0}_n & & -1 & 0 & 1 & 0 \\ \hline
 & \mathbf{0}_n & & -1 & \mathbf{0}_{n-2} & 0 & 0 & 0 & 1 & 1 \\ \hline
 & & & & & & & & & \\
 & 0 & & & M & & \multicolumn{4}{c}{0} \\
 & & & & & & & & & \\ \hline
 & & & & \mathbf{0}_n & & 1 & -1 & -1 & 1 \\
 & 0 & & & (-1)_n & & -1 & 1 & 1 & 0 \\
 & & & & \mathbf{0}_n & & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}
$$

After row-reduction, blocks $(1,1)$ and $(2,1)$ yield an identity matrix, which, given the zeros on blocks $(3,1)$, $(4,1)$, and $(5,1)$, means that we have maximal rank there.

Now consider blocks $(3,2)$ and $(4,2)$. We see that we can add the top row of this double-block to the one right below to get a 1 cancelled out. After that we can add this modified row to the one right below, producing the same effect. In this way blocks $(3,2)$ and $(4,2)$ are row-reduced to a negative identity. Now, these row-reductions affect also block $(4,3)$.

We get

$$\begin{array}{r|c}
 & \begin{array}{ccc|ccc|cccc} y_1 & \ldots & y_n & u_1 & \ldots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r} M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ N_n - L_1 \\ \hline M_n - N_1 \\ N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2 \end{array} &
\left(\begin{array}{c|c|cccc}
I_n & 0 & & ? & ? & \\
\hline
0 & -I_n & (\mathbf{0}_n)^T & (\mathbf{0}_n)^T & (\mathbf{1}_n)^T & (\mathbf{1}_n)^T \\
\hline
 & \mathbf{0}_n & 1 & -1 & -1 & 1 \\
0 & (-1)_n & -1 & 1 & 1 & 0 \\
 & \mathbf{0}_n & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}$$

We now subtract all the rows of the second row-block from row $L_2 - K_1$. We also add row $K_1 - K_2$ to $L_2 - K_1$, and subtract row $K_1 - K_2$ from $L_1 - L_2$. We get

$$\begin{array}{r|c}
 & \begin{array}{ccc|ccc|cccc} y_1 & \ldots & y_n & u_1 & \ldots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r} M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ N_n - L_1 \\ \hline M_n - N_1 \\ N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2 \end{array} &
\left(\begin{array}{c|c|cccc}
I_n & 0 & & ? & ? & \\
\hline
0 & -I_n & (\mathbf{0}_n)^T & (\mathbf{0}_n)^T & (\mathbf{1}_n)^T & (\mathbf{1}_n)^T \\
\hline
 & & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1-n & -n \\
 & & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}$$

Finally, adding (n times) row $L_1 - L_2$ to $L_2 - K_1$ and rescaling row $L_2 - K_1$ by -1, we get

$$
\begin{array}{rc}
 & \begin{array}{ccc|ccc|cccc} y_1 & \dots & y_n & u_1 & \dots & u_n & v_1 & v_2 & w_1 & w_2 \end{array} \\
\begin{array}{r}
M_1 - M_2 \\ \vdots \\ M_{n-1} - M_n \\ N_n - L_1 \\ \hline
M_n - N_1 \\ N_1 - N_2 \\ \vdots \\ N_{n-1} - N_n \\ \hline
L_1 - L_2 \\ L_2 - K_1 \\ K_1 - K_2
\end{array}
&
\left(\begin{array}{c|c|cccc}
 & & & & & \\
I_n & 0 & & ? & ? & \\
 & & & & & \\
 & & & & & \\ \hline
 & & & & & \\
 & & & & & \\
0 & -I_n & (\mathbf{0}_n)^T & (\mathbf{0}_n)^T & (\mathbf{1}_n)^T & (\mathbf{1}_n)^T \\
 & & & & & \\ \hline
 & & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 2n-1 & 0 \\
 & & 1 & -1 & 0 & 0
\end{array}\right)
\end{array}
$$

We can see that this matrix has full rank $(2n+3)$ over a field with characteristic $p \nmid (2n-1)$ and that the previous to last row becomes a row of all zeroes over a field with characteristic $p \mid (2n-1)$. So, since G_n has $(2n+4)$ vertices, we get

$$wcdim(G_n, \mathbb{F}) = |V(G_n)| - rank(A_{G_n}) = (2n+4) - (2n+3) = 1 \text{ if } char(\mathbb{F}) \nmid (2n-1),$$

and

$$wcdim(G_n, \mathbb{F}) = |V(G_n)| - rank(A_{G_n}) = (2n+4) - (2n+2) = 2 \text{ if } char(\mathbb{F}) \mid (2n-1).$$

which is what we wanted to show. □

Corollary 1. Let p be a prime. For every $n \geq p+5$ there is a graph G of order n for which $wcdim(G, \mathbb{F})$ takes different values depending on whether $char(\mathbb{F})$ is p or not.

Proof. This is an immediate corollary of Lemma 1 and Theorems 1 and 2. □

The graphs constructed in the previous theorem and corollary are the smallest known graphs to have characteristic-dependent well-covered dimension for any given characteristic. Previously, the smallest known graphs with this property that could be constructed were crown graphs as described in Birnbaum et al.[1].

Based upon the results generated from our Sage script, we submit as a conjecture that these graphs constructed be the smallest graphs (by order) that have characteristic-dependent well-covered dimension for any given characteristic. We have only verified this to be the case (using Sage) for graphs up to order 10. We believe that this is also a very interesting open problem.

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